

Localization & Mitigation of Cascading Failures in Power Systems, Part II: Localization

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Abstract—Cascading failures in power systems propagate non-locally, making the control and mitigation of outages hard. In Part II of this paper, we continue the study of tree partitioning of transmission networks and characterize analytically line failure localizability. We show that a tree-partition region can be further decomposed into disjoint cells in which line failures will be contained. When a non-cut set of lines are tripped simultaneously, its impact is localized within each cell that contains a line outage. In contrast, when a bridge line that connects two tree-partition regions is tripped, its impact propagates globally across the network, affecting the power flows on all remaining lines. This characterization suggests that it is possible to improve system reliability by switching off certain transmission lines to create more, smaller cells, thus localizing line failures and reducing the risk of large-scale outages. We demonstrate using the IEEE 118-bus test system that switching off a negligible portion of lines allows the impact of line failures to be significantly more localized without substantial changes in line congestion.

I. INTRODUCTION

In Part I of this paper [1], we establish the spectral representation of power redistribution that precisely captures the Kirchhoff's Laws in terms of the distribution of different families of subtrees in the transmission network. This new representation motivates a novel approach to localize cascading failures in power systems.

Contributions of Part II of this paper: *We prove that tree partitioning proposed in Part I provides a precise analytical characterization of line failure localizability, and we show how to use this characterization to mitigate failure cascading by switching off a small number of carefully chosen transmission lines and creating in this way a finer network tree partition.* Our results not only rigorously establish the intuition of the community that failures cannot cross bridges, but also reveal a less intuitive finer-grained concept, called cells (defined by partitioning the network using its cut vertices, see Section IV) inside each tree-partition region, that encodes more precise information on failure propagation. In particular, we prove that: (a) a non-cut failure in a cell will only impact power flows on transmission lines within that cell, regardless of whether the failure involves a single line or multiple lines *simultaneously*; and conversely (b) a non-cut failure in a cell will almost surely impact flows on *every* remaining line within that cell. This is not intuitive to our best knowledge. Our results also demonstrate how the impact of cut failures propagate globally in a way that depends on the design of balancing rules and a network's

topological structure. This work builds on the recent work on the line outage distribution factor, e.g., [2], [3], and shows that tree partitioning is a particularly useful representation of this factor, one that captures many aspects of how line failures can cascade.

The formal characterization of localizability in the case of a single line failure is given in Theorem 2 of Section III, which summarizes the technical results in Sections IV and V. In particular, in Section IV, we characterize power redistribution after the tripping of a single non-bridge line and show that the impact of such a failure only propagates within its own cell. In Section V, we consider the failure of a single bridge line and prove that such a failure propagates globally across the network and impacts the power flows on all transmission lines.

In Section VI, we extend Theorem 2 to the case of multiple simultaneous line failures, and show that the aggregate impact of such failure consists of two parts: (a) a part from power redistribution that generalizes the case of a single non-bridge failure, and such impact can be decomposed in accordance to the cells where the failures occur; (b) a part from power balancing rule that generalizes the case of a single bridge failure and captures how the system handles disconnected components. These results rely on properties of tree partitioning proved in Part I [1]. In particular, we make use of the block decomposition of tree-partition regions to completely eliminate simple loops containing edges among distinct cells. The Simple Loop Criterion in Part I [1] then guarantees failure localization. Lastly, we apply classical techniques from algebraic geometry to address potential pathological system specifications for our converse results.

The characterizations we provide in Theorem 2 and Section VI suggest a new approach for mitigating the impact of cascading failures by switching off certain transmission lines to create smaller regions/cells. This better localizes line failures and reduces the risk of large-scale blackouts. In Section VII, we illustrate this approach using the IEEE 118-bus test network. We demonstrate that switching off only a negligible portion of transmission lines can lead to significantly better control of cascading failures, and more importantly, without significant increase in line congestion across the grid.

Finally we remark that tree partitioning can be useful for planning applications. It can also be used as an emergency measure the same way controlled islanding (see e.g. [4]–[12]) is used to prevent large-scale blackouts¹. For instance, tree partitions can be pre-computed offline for different contingencies and loading conditions, and implemented as soon as a contingency is detected using special protection schemes. It would be interesting to understand the tradeoffs of post-contingency corrective tree partitioning and controlled islanding and how they can be synergistically integrated for

This work has been supported by Resnick Fellowship, Linde Institute Research Award, NWO Rubicon grant 680.50.1529, NSF through grants CCF 1637598, ECCS 1619352, ECCS 1931662, CNS 1545096, CNS 1518941, CPS ECCS 1739355, CPS 154471.

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¹We thank Janusz Bialek for suggesting this as a potential application.

failure mitigation. In this paper, we focus on proving general failure localization properties of tree partitioning and leave its application for failure mitigation and network planning for future work.

II. PRELIMINARIES

In this section, we present the cascading failure model that our main technical results build upon. This model extends the DC power flow model presented in Part I to multiple stages, and is a special case of the integrated failure propagation model that we introduce in Part III.

Given a power network, we describe the cascading failure process by keeping track of the set of failed lines at different stages indexed by $\mathfrak{N} := \{1, 2, \dots, N\}$. Each stage $n \in \mathfrak{N}$ corresponds to a graph $\mathcal{G}(n) := (\mathcal{N}, \mathcal{E} \setminus \mathcal{B}(n))$, where $\mathcal{B}(n)$ is the set of all tripped lines at stage n and is naturally nested:

$$\mathcal{B}(n) \subset \mathcal{B}(n+1), \quad \forall n \in \mathfrak{N}.$$

Within each stage n , the power flow redistributes over the network described by $\mathcal{G}(n)$ according to the DC power flow model (see Part I for more details). If all the branch flows are below the corresponding line ratings, then the new operating point is secure and the cascade stops. Otherwise, let $\mathcal{F}(n)$ be the subset of lines whose branch flows exceed the corresponding line ratings. The lines in $\mathcal{F}(n)$ are tripped in stage n , i.e., $\mathcal{B}(n+1) = \mathcal{B}(n) \cup \mathcal{F}(n)$. This process repeats for stage $n+1$ and so on.

Next, we focus on a fixed stage $n \in \mathfrak{N}$, and describe how a single line failure impacts the branch flows on remaining lines. To simplify the notation, we drop the stage index from symbols such as $\mathcal{G}(n)$ and simply write them as \mathcal{G} . Recall that when a line e is tripped from \mathcal{G} , if the newly formed graph $\mathcal{G}' := (\mathcal{N}, \mathcal{E} \setminus \{e\})$ is still connected, then the branch flow change on a line \hat{e} with the same power injections p is given by $\Delta f_{\hat{e}} = K_{\hat{e}e} f_e$, where $K_{\hat{e}e}$ is the line outage distribution factor (LODF) [13] from e to \hat{e} . In Part I of this paper, by studying the transmission network Laplacian matrix, we derive a new formula for $K_{\hat{e}e}$ that precisely captures the Kirchhoff's Laws in terms of graphical structures. The new formula is replicated here for reference:

Theorem 1. Let $e = (i, j), \hat{e} = (w, z) \in \mathcal{E}$ be edges such that $\mathcal{G}' = (\mathcal{N}, \mathcal{E} \setminus \{e\})$ is connected. Then, $K_{\hat{e}e}$ is given by

$$B_{\hat{e}} \times \frac{\sum_{E \in \mathcal{T}(\{i,w\}, \{j,z\})} \chi(E) - \sum_{E \in \mathcal{T}(\{i,z\}, \{j,w\})} \chi(E)}{\sum_{E \in \mathcal{T}_{\mathcal{E} \setminus \{i,j\}}} \chi(E)}.$$

Theorem 1 allows us to check algebraic properties of $K_{\hat{e}e}$ by examining the network topology \mathcal{G} . Recall $K_{\hat{e}e} \neq 0$ means that the failure of e can potentially lead to the failure of \hat{e} . In Part I, we proved a *Simple Loop Criterion*: $K_{\hat{e}e} \neq 0$ “if” and only if there exists a simple loop containing both e and \hat{e} in \mathcal{G} . In particular, for two edges e and \hat{e} such that e is not a bridge, if there is no simple loop containing both e and \hat{e} , then $K_{\hat{e}e} = 0$.

If the post-contingency graph \mathcal{G}' is disconnected, then it is possible that the original injections p are no longer balanced in some connected components of \mathcal{G}' . Thus, to compute the new power flows, a power balancing rule \mathcal{R} needs to be applied. Several such rules have been proposed and evaluated in the literature based on load shedding or generator response [2], [14]–[17]. In this work, we do not specialize to any such rule but, instead, we identify key properties of these rules that guarantee our results to hold. For this reason, our more

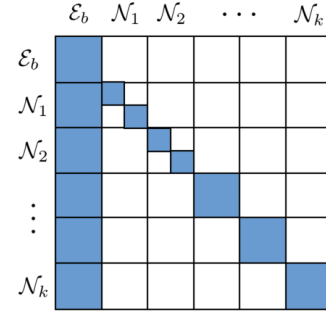


Fig. 1. Non-zero entries of the $K_{\hat{e}e}$ matrix (as represented by the dark blocks) for a graph with tree-partition $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$ and bridge set \mathcal{E}_b . The small blocks inside \mathcal{N}_1 and \mathcal{N}_2 represent cells inside those regions.

abstract approach allows us to characterize the power flow redistribution under a broad class of power balancing rules.

III. SINGLE LINE FAILURE

In this section we state our main analytical result in the case of a single-line failure. This is the critical building block for characterizing the case of multi-line failures, to be presented in Section VI.

Our result applies to scenarios in which the system is operating under *normal* conditions, i.e., when the following two assumptions are satisfied: (a) the injections are *island-free* (see Definition 7); and (b) the grid is *participating* with respect to its power balancing rule (see Definition 9). Moreover, to avoid pathological cases, we add a small perturbation to the line susceptances drawn from a probability measure μ , which we assume to be absolutely continuous with respect to the Lebesgue measure \mathcal{L}_m on \mathbb{R}^m (see Section V).

Theorem 2. For a power network operating under normal conditions, $K_{\hat{e}e} \neq 0$ “if” and only if:

- 1) e, \hat{e} are within the same tree-partition region and e, \hat{e} belong to the same cell; or
- 2) e is a bridge.

The “if” part should be interpreted as an almost sure event in μ (see Definition 6). The conditions stated in Theorem 2 are far from being restrictive, but satisfied in most practical settings. More specifically, we emphasize that: (a) μ being absolutely continuous with respect to \mathcal{L}_m holds in almost all relevant stochastic models for such perturbations (see Section IV); and (b) the conditions that the injections are island-free and the grid is participating are satisfied in typical operating scenarios of power systems (see Section V). This result implies that, for essentially all networks of interest, the tree partition encodes rich information on how the failure of a line propagates through the network.

The findings of Theorem 2 are visualized in Fig. 1, where it becomes clear how the tree partition is linked to the sparsity of the $K_{\hat{e}e}$ matrix. It suggests that for a fully meshed transmission network with a trivial tree partition consisting of a single region/cell, it can be beneficial to switch off certain lines so that more tree-partition regions or cells are created, localizing in this way the line failure within the region/cell in which it occurs. We study this network planning and design opportunity in Section VII.

The next two sections are devoted to the proof of Theorem 2. We first characterize power redistribution after the tripping

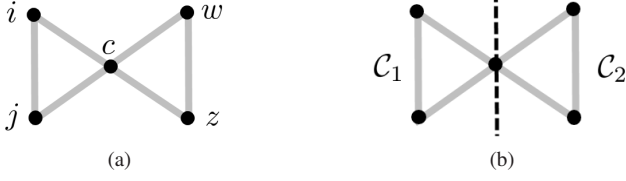


Fig. 2. (a) A butterfly network. (b) The block decomposition of the butterfly network into cells C_1 and C_2 .

of a single non-bridge line in Section IV, and then consider in Section V the failure of a single bridge. In Section VI, we generalize our single-line failure characterization to the case of multi-line failures.

IV. NON-BRIDGE FAILURES ARE LOCALIZABLE

In this section, we show that the impacts of a single non-bridge line failure are localized within a tree-partition region. Recall from Proposition 10 in Part I that $K_{\hat{e}e}$ is a precise indicator on whether the failure e can potentially lead to the successive failure of line \hat{e} .

A. Impact across Regions

We consider first the case where \hat{e} does not belong to the same region as e , that is, \hat{e} either belongs to a different region or \hat{e} is a bridge.

Proposition 3. *Consider a power network \mathcal{G} with a tree partition $\mathcal{P} = \{N_1, N_2, \dots, N_k\}$. Let $e, \hat{e} \in \mathcal{E}$ be two different edges such that e is not a bridge. Then $K_{\hat{e}e} = 0$ for any \hat{e} that is not in the same region containing e .*

See Appendix I for a proof. This result implies that, when a non-bridge line e fails, any line \hat{e} not in the same tree-partition region as e will not be affected, regardless of whether \hat{e} is a bridge. In other words, non-bridge failures cannot propagate through the boundaries formed by the tree-partition regions of \mathcal{G} .

B. Impact within Regions

It is reasonable to expect that the converse to the above result is also true. That is, if e, \hat{e} belong to the same region (and thus e is not a bridge), we would expect $K_{\hat{e}e} \neq 0$. This, however, might not always be the case for two reasons: 1) some nodes within a tree-partition region may “block” simple loops containing both e and \hat{e} ; 2) the graph \mathcal{G} may be too symmetric. We elaborate on these two scenarios separately in the following two subsections.

1) *Block Decomposition:* The following example illustrates the situation in which nodes inside a tree-partition region may “block” simple loops containing both e and \hat{e} .

Example 1. *Consider the butterfly network shown in Fig. 2(a) and pick $e = (i, j)$ and $\hat{e} = (w, z)$. It is not hard to see that any loop containing e and \hat{e} must pass through the central node c at least twice and hence is never simple. The Simple Loop Criterion (Theorem 11 in [1]) immediately yields that $K_{\hat{e}e} = 0$.*

The underlying issue in this example is that the butterfly network is not 2-connected, which means that the removal of a single node (in this case the central node c) can disconnect the original graph. Network nodes with such a property are

referred to as *cut vertices* in graph theory. As in Example 1, cut vertices prevent the existence of global simple loops.

Fortunately, we can precisely capture such an effect by decomposing each tree-partition region further through the classical *block decomposition* [18]. Recall that the block decomposition of a graph is a partition of its edge set \mathcal{E} such that each partitioned component is 2-connected, see Fig. 2(b) for an illustration. The block decomposition of a graph always exists and can be found in linear time [19]. We refer to the components of this decomposition as *cells* to reflect the fact that they are smaller parts within a tree-partition region. Note that two different cells within the same tree-partition region may share a common vertex, since the block decomposition induces a partition only of the network edges.

Lemma 4. *If e, \hat{e} are two distinct edges within the same tree-partition region that belong to different cells, then $K_{\hat{e}e} = 0$.*

Proof. Let C_e and $C_{\hat{e}}$ be the two cells to which e and \hat{e} respectively belong. It is a well-known result that any path from a node in C_e to a node in $C_{\hat{e}}$ must pass through a common cut vertex in C_e or $C_{\hat{e}}$ [18]. It is then impossible to find a simple loop containing both e and \hat{e} and, thus, the Simple Loop Criterion implies that $K_{\hat{e}e} = 0$. \square

2) *Symmetry:* Next, we show how graph symmetries (more specifically, graph automorphisms) may also block the propagation of failures. We first illustrate the issue by means of an example.

Example 2. *Consider the power network \mathcal{G} consisting of the complete graph on n vertices ($n \geq 4$) and line susceptances all equal to 1. Pick two edges $e = (i, j)$ and $\hat{e} = (w, z)$ that do not share any common endpoints. By symmetry, it is easy to see that there is a bijective correspondence between $\mathcal{T}(\{i, w\}, \{j, z\})$ and $\mathcal{T}(\{i, z\}, \{j, w\})$, and thus*

$$\sum_{E \in \mathcal{T}(\{i, w\}, \{j, z\})} \chi(E) - \sum_{E \in \mathcal{T}(\{i, z\}, \{j, w\})} \chi(E) = 0.$$

By Theorem 1, we then have $K_{\hat{e}e} = 0$.

A complete graph is 2-connected and thus consists of a single cell. Example 2 shows that even if two edges e, \hat{e} are within the same cell, when the graph \mathcal{G} is rich in symmetry, it is still possible that a failure of line e does not impact another line \hat{e} . Nevertheless, this issue is not critical as such a symmetry almost never happens in practical systems, because of heterogeneity in line susceptances. In fact, even if the power network topology is highly symmetric, an infinitesimal change on the line susceptances is enough to break the symmetry, as we now show.

More specifically, we take the line susceptances to be average values B_e plus random perturbations $\omega = (\omega_e : e \in \mathcal{E})$ drawn from a probability measure μ . Such perturbations can come from manufacturing error or measurement noise. The perturbed system² shares the same topology (and thus tree partition) as the original system, yet admits perturbed susceptances $B + \omega$. The randomness of ω implies the factors $K_{\hat{e}e}$ are now random variables. Recall that we assume μ to be *absolutely continuous* with respect to Lebesgue measure \mathcal{L}_m on \mathbb{R}^m , which means that for any measurable set S such that $\mathcal{L}_m(S) = 0$, we have $\mu(S) = 0$.

²We assume the perturbation ensures $B_e + \omega_e > 0$ for any $e \in \mathcal{E}$ so that the new susceptance is physically meaningful.

Proposition 5. Consider a power network \mathcal{G} with susceptibility perturbation μ and let e, \hat{e} be two distinct edges within the same cell. If μ is absolutely continuous with respect to \mathcal{L}_m , then $K_{\hat{e}e} \neq 0$ μ -almost surely, i.e., $\mu(K_{\hat{e}e} \neq 0) = 1$.

See Appendix II for a proof. Note that, by the Radon-Nikodym Theorem [20], the probability measure μ is absolutely continuous with respect to \mathcal{L}_m if and only if it affords a probability density function. This essentially amounts to requiring the measure μ to not contain Dirac masses. As a result, we see that for almost all relevant stochastic models on such perturbations (e.g., truncated Gaussian noise with arbitrary covariance, bounded uniform distribution, truncated Laplace distribution), the assumption applies and, thus, $K_{\hat{e}e} \neq 0$ for e, \hat{e} within the same cell almost surely, no matter how small the perturbation is.³

This perturbation approach is also useful for our results in the following sections. When we take this approach, our result often constitutes two directions: (a) an “only if” direction that should be interpreted as normal; and (b) an “if” direction that holds almost surely in μ . To simplify the presentation, we henceforth fix an absolutely continuous perturbation μ with respect to \mathcal{L}_m , and define the following:

Definition 6. For two predicates p and q , we say that p “if” and only if q when both of the following hold:

$$p \Rightarrow q, \quad q \Rightarrow \mu(p) = 1.$$

We say the “if” is in μ -sense when we need to emphasize that the “if” direction only holds almost surely in μ .

Proposition 3, Lemma 4, and Proposition 5 prove the first half of Theorem 2.

V. BRIDGE FAILURES PROPAGATE

The remaining case necessary to prove Theorem 2 is a characterization of power flow redistribution when a single bridge is tripped. Here, we show that such failures generally propagate through the entire network.

A. Extended $K_{\hat{e}e}$ and Island-free Operation

When e is a bridge, tripping e disconnects the power grid into two islands, and the power in each connected component may not be balanced. Such power imbalance can be resolved by a power balancing rule \mathcal{R} (see [14]–[17] for examples of such rules), which together with the DC model uniquely determines the post-contingency branch flows (and thus the branch flow changes $\Delta f_{\hat{e}}$). For the purpose of unified notation, we extend the definition of $K_{\hat{e}e}$ through $K_{\hat{e}e} := \Delta f_{\hat{e}}/f_e$ to the case where e is a bridge. Besides being related to the B and C matrices, this extended $K_{\hat{e}e}$ factor also depends on the power injections p and the power balance rule \mathcal{R} . It is only meaningful if $f_e \neq 0$, which is not a restriction since when $f_e = 0$ we clearly have $\Delta f_{\hat{e}} = 0$ for all remaining lines.

Definition 7. For a power network \mathcal{G} , an injection vector p is said to be **island-free** if the resulting branch flow f satisfies $f_e \neq 0$ for every bridge e .

Intuitively, island-free means that no part of the grid is self-balanced. That is, the aggregate power injection over each of the tree-partition regions is non-zero and hence there

must be power exchanges among the regions. For an island-free grid, every bridge carries nonzero branch flow and thus the extended $K_{\hat{e}e}$ factors are always well-defined.

B. Participating Buses

Suppose a bridge e of a power network \mathcal{G} is tripped and separates the network into two islands. Consider one of these islands $\mathcal{D} := (\mathcal{N}_{\mathcal{D}}, \mathcal{E}_{\mathcal{D}})$ and let u be the endpoint of e that belongs to \mathcal{D} . Note \mathcal{D} may contain multiple tree-partition regions of the original network. Tripping e from the grid leads to a power imbalance f_e (with a sign determined by the original flow direction on e) and the balancing rule \mathcal{R} distributes such imbalance to a set of *participating* buses from \mathcal{D} so that the total power imbalance f_e is canceled out.

The rules studied in the literature, e.g., [2], [14]–[17], are typically *linear* in the sense that, for any participating bus j , the injection adjustment dictated by the rule \mathcal{R} is linear in f_e . Given a rule \mathcal{R} , denote by $\mathcal{N}_{\mathcal{R}}$ the set of participating buses in \mathcal{D} and let $n_r = |\mathcal{N}_{\mathcal{R}}|$. The rule \mathcal{R} can then be interpreted as a linear transformation from \mathbb{R} to \mathbb{R}^{n_r} given by $\mathcal{R}(f_e) := (\alpha_j f_e : j \in \mathcal{N}_{\mathcal{R}})$, where $\alpha_j > 0$, $\sum_j \alpha_j = 1$, and f_e is the total power imbalance in \mathcal{D} after the bridge e failure. Note that f_e could be both positive or negative, depending on whether the island has an excess or a shortfall of power.

Different rules correspond to different choices of participating buses and coefficients α_j ’s. For instance, if the generators in \mathcal{D} are uniformly adjusted to compensate the imbalance [2], [14], then $\mathcal{N}_{\mathcal{R}}$ coincides with the set of generators in \mathcal{D} and $\alpha_j = 1/|\mathcal{N}_{\mathcal{R}}|$. As another example, if the imbalance is regulated through Automatic Generation Control (AGC), then $\mathcal{N}_{\mathcal{R}}$ is the set of controllable generators and α_j are the normalized generator *participation factors*.

Denote the injection adjustment vector over all buses in \mathcal{D} (that results from tripping e and the power balancing by \mathcal{R}) by $\Delta p_{\mathcal{D}}$. Let u be the endpoint of e in \mathcal{D} and denote by $C_{\mathcal{D}}$, $L_{\mathcal{D}}$ the incidence and Laplacian matrix of the island \mathcal{D} , respectively. Put $B_{\mathcal{D}}$ to be the sub-matrix of B corresponding to edges in \mathcal{D} and let $A_{\mathcal{D}}$ be the extended inverse of $L_{\mathcal{D}}$ (see Part I for more details). Then the branch flow changes on the remaining lines in \mathcal{D} are given by

$$\Delta f_{\mathcal{D}} = B_{\mathcal{D}} C_{\mathcal{D}}^T A_{\mathcal{D}} \Delta p_{\mathcal{D}}.$$

We now determine when $(\Delta f_{\mathcal{D}})_{\hat{e}} = 0$ for a remaining line $\hat{e} \in \mathcal{E}_{\mathcal{D}}$, which in turn characterizes whether the extended $K_{\hat{e}e}$ is zero or not.

Proposition 8. For $\hat{e} \in \mathcal{E}_{\mathcal{D}}$, we have $\Delta f_{\hat{e}} \neq 0$ “if” and only if there exists $j \in \mathcal{N}_{\mathcal{R}}$ such that there is a simple path in \mathcal{D} from u to j containing \hat{e} .

The “if” part in this result is in μ -sense as discussed in Section IV. See Appendix III for a proof.

Proposition 8 shows that the positions of participating buses in \mathcal{D} play an important role in distributing the power imbalance across the network. In particular, the power balancing rule \mathcal{R} changes the branch flow on every edge that lies in a path from the failure point u to the set of participating buses. As a result, if \hat{e} is a bridge connecting two tree-partition regions \mathcal{D}_1 and \mathcal{D}_2 in \mathcal{D} , assuming $u \in \mathcal{D}_1$, then $\Delta f_{\hat{e}} \neq 0$ “if” and only if \mathcal{D}_2 contains a participating bus (since a path from u to any node in \mathcal{D}_2 must pass through \hat{e}). If \hat{e} is not a bridge, then we can devise a simple sufficient

³Our results can be readily extended to the case when μ contains Dirac masses. See Appendix X for more discussion.

condition for $\Delta f_{\hat{e}} \neq 0$ using *participating regions*, defined as follows:

Definition 9. Consider a power network \mathcal{G} with tree partition $\mathcal{P} = \{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$ operating under power balancing rule \mathcal{R} with a set $\mathcal{N}_{\mathcal{R}}$ of participating buses. A region \mathcal{N}_i with block decomposition $\{\mathcal{C}_1^i, \mathcal{C}_2^i, \dots, \mathcal{C}_{m_i}^i\}$ is said to be a **participating region** if $\mathcal{N}_{\mathcal{R}} \cap \mathcal{C}_j^i$ contains a non-cut vertex for $j = 1, 2, \dots, m_i$. \mathcal{G} is said to be a **participating grid** if \mathcal{N}_i is participating for $i = 1, 2, \dots, k$.

If a power network does not contain single-point vulnerabilities such as cut vertices within each tree-partition region, then the tree-partition regions are identical with cells. In this case, a region is participating if it contains at least one bus that participates in power balancing and is not the endpoint of a bridge, which is often the case. As such, it is reasonable to assume most tree-partition regions are participating regions and hence most power grids are participating.

The following result shows that a region being participating implies all edges in this region are impacted when the original bridge e is tripped (its proof is presented in Appendix IV).

Lemma 10. If \mathcal{N}_i is a participating region and $f_e \neq 0$, then $\Delta f_{\hat{e}} \neq 0$ μ -almost surely for any $\hat{e} \in \mathcal{N}_i$, i.e., $\mu(\Delta f_{\hat{e}} \neq 0) = 1$.

Given the above, we now state our main result for bridge failures.

Proposition 11. Consider a participating grid \mathcal{G} with island-free injections p . If e is a bridge of \mathcal{G} , then for any $\hat{e} \neq e$, $K_{\hat{e}e} \neq 0$ μ -almost surely, i.e., $\mu(K_{\hat{e}e} \neq 0) = 1$.

This proposition proves the second part of Theorem 2.

VI. MULTI-LINE FAILURES

In this section, we generalize our results in Sections IV and V to the case of multi-line failures. That is, we characterize the branch flow changes on remaining lines after a set E of lines are simultaneously removed from service.

Similar to a single-line outage, the impact of tripping a set E of lines propagates differently depending on whether the post-contingency graph $\mathcal{G}' := (\mathcal{N}, \mathcal{E} \setminus E)$ remains connected.

Definition 12. A subset of edges $E \subset \mathcal{E}$ is said to be a **cut set** if the graph \mathcal{G}' after removing E is not connected, or a **non-cut set** if it is not a cut set.

A. Non-cut Failure

In this subsection, we focus on the case where E is a non-cut set. Partition the column vector $f = [f_E; f_{-E}]$, where f_E and f_{-E} are the pre-contingency power flows in E and in $\mathcal{E} \setminus E$ respectively. Let \hat{f}_{-E} be the post-contingency line flows in $\mathcal{E} \setminus E$ and $\hat{\theta}$ be the post-contingency phase angles. We study the power flow change $\Delta f_{-E} := \hat{f}_{-E} - f_{-E}$ on the remaining lines in terms of the pre-contingency power flows f_E on the simultaneously outaged lines in E . Even though the DC power flow model is linear, the impacts $\Delta f_{\hat{e}}$ on remaining lines \hat{e} are generally not the sum of the impacts from tripping lines in E separately (i.e., $\Delta f_{\hat{e}} \neq \sum_{e \in E} K_{\hat{e}e} f_e$ in general).

Without loss of generality, we partition the matrices (B, C) into submatrices corresponding to the outaged lines

in E and the remaining lines in $\mathcal{E} \setminus E$, i.e.:

$$B = \text{diag}(B_E, B_{-E}), \quad C = [C_E, C_{-E}].$$

Then the DC power flow equations that describe the post-contingency network are given by

$$p = C_{-E} \hat{f}_{-E} \quad (1a)$$

$$\hat{f}_{-E} = B_{-E} C_{-E}^T \hat{\theta}. \quad (1b)$$

From this, we can rewrite the power flow changes Δf_{-E} on the remaining lines in terms of pre-contingency flows f_E on the outaged lines [21]:

$$\Delta f_{-E} = \underbrace{B_{-E} C_{-E}^T A C_E (I - B_E C_E^T A C_E)^{-1}}_{K^E} f_E, \quad (2)$$

where A is defined as in Part I. The $|\mathcal{E} \setminus E| \times |E|$ matrix K^E in (2) is known as the generalized line outage distribution factor (GLODF),⁴ which generalizes the LODF discussed in previous sections from single-line outages ($|E| = 1$) to multi-line outages ($|E| \geq 1$). More specifically, for each remaining line $\hat{e} \in \mathcal{E} \setminus E$, let $K_{\hat{e}E}$ denote the \hat{e} -th row of K^E , then the power flow change on line \hat{e} depends on the pre-contingency flows on *all* lines $e \in E$ according to the \hat{e} -th row of (2):

$$\Delta f_{\hat{e}} = K_{\hat{e}E} f_E = \sum_{e \in E} K_{\hat{e}e}^E f_e. \quad (3)$$

For a single-line failure $E = \{e\}$, $K_{\hat{e}E}$ reduces to the LODF $K_{\hat{e}e}$. We remark that when $|E| > 1$, $K_{\hat{e}E}^E$ is generally different from $K_{\hat{e}e}$ (see equation (7) below).

The next result describes the localization property of tree partition in the case of a multi-line failure.

Theorem 13. Let $K_{\hat{e}e}^E$ be the (\hat{e}, e) -th entry of the GLODF K^E . Then $K_{\hat{e}e}^E \neq 0$ “if” and only if e and \hat{e} are within the same cell.

This result generalizes the non-bridge failure case of Theorem 2 and shows that such failures are localized within the corresponding cells (in one stage). In particular, similar to the single-line failure case, if \hat{e} does not share a cell with any line $e \in E$ then its line flow will not be impacted. However, unlike the single-line failure case where $K_{\hat{e}e} \neq 0$ implies $f_{\hat{e}} \neq 0$, we do not necessarily have $\Delta f_{\hat{e}} \neq 0$ even when \hat{e} is in the same cell as some $e \in E$. This is because the impact on \hat{e} from different line failures in E can potentially cancel each other. Nevertheless, it can be shown that under mild conditions on the topology of \mathcal{G} or by adding a perturbation to the system injection p (similar to how we perturb the line susceptances B with μ), such a cancellation almost surely does not happen. Theorem 13 is illustrated in the following example.

Example 3. Consider an $N - 2$ event where lines e_1 and e_2 are simultaneously tripped, i.e., $E := \{e_1, e_2\}$. The power flow change on a remaining line \hat{e} is

$$\Delta f_{\hat{e}} = K_{\hat{e}e_1}^E f_{e_1} + K_{\hat{e}e_2}^E f_{e_2},$$

⁴ The formula of K^E can be derived by considering the pre-contingency network with changes Δp in power injections that are judiciously chosen to emulate the effect of simultaneous line outages in E . The reference [21] seems to be the first to introduce the use of matrix inversion lemma to study the impact of network changes on line currents in power systems. This method is also used in [22] to derive the GLODF for ranking contingencies in security analysis. The GLODF has also been derived earlier, e.g., in [23], and re-derived recently in [24], [25], without the simplification of the matrix inversion lemma.

where $K_{\hat{e}e_i}^E$ is the (\hat{e}, e_i) -th entry of K^E . There are two cases:

- 1) Lines e_1, e_2 are in the same cell: $E \subset \mathcal{C}$. According to Theorem 13, we have

$$\Delta f_{\hat{e}} = \begin{cases} 0, & \text{if } \hat{e} \notin \mathcal{C} \\ K_{\hat{e}e_1}^E f_{e_1} + K_{\hat{e}e_2}^E f_{e_2}, & \text{if } \hat{e} \in \mathcal{C}. \end{cases}$$

- 2) Lines e_1, e_2 are in different cells: $e_i \in \mathcal{C}_i$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Then

$$\Delta f_{\hat{e}} = \begin{cases} 0, & \text{if } \hat{e} \notin \mathcal{C}_1 \cup \mathcal{C}_2, \\ K_{\hat{e}e_1}^E f_{e_1}, & \text{if } \hat{e} \in \mathcal{C}_1, \\ K_{\hat{e}e_2}^E f_{e_2}, & \text{if } \hat{e} \in \mathcal{C}_2. \end{cases}$$

In the rest of this subsection, we look at topological structures of the transmission network \mathcal{G} that underlie how multi-line failures in E interact to produce the aggregate impact Δf_{-E} . It also serves as the background for the proof of Theorem 13 in Appendix VII.

Recall from Part I that $D_{\hat{e},e}$ represents the generation shift sensitivity factor from e to \hat{e} . Let $D_{-E,E}$ be an $|\mathcal{E} \setminus E| \times |E|$ matrix whose (\hat{e}, e) -th element is $D_{\hat{e},e}$. It is easy to show that $D_{-E,E} = B_{-E} C_{-E}^T A C_E$. Then (2) can be rewritten equivalently as

$$\tilde{f}_E = (I - B_E C_E^T A C_E)^{-1} f_E, \quad (4a)$$

$$\Delta f_{-E} = B_{-E} C_{-E}^T A C_E \tilde{f}_E = D_{-E,E} \tilde{f}_E. \quad (4b)$$

Equations (4) interpret the impact on the remaining lines of tripping E simultaneously as consisting of two steps. First, the pre-contingency flows f_E “mix” into a vector of flow changes \tilde{f}_E , and second, the mixed flow changes \tilde{f}_E propagate linearly to the remaining edges via (4b) to produce the post-contingency impact: i.e., $\Delta f_{\hat{e}} = \sum_{e \in E} D_{\hat{e},e} \tilde{f}_e$ on each remaining line \hat{e} .

Noting that $K_{\hat{e}e} = D_{\hat{e},e}/(1 - D_{e,e})$ when e is not a bridge [13], we can write the flow change on a remaining line \hat{e} as

$$\Delta f_{\hat{e}} = \sum_{e \in E} K_{\hat{e}e} (1 - D_{e,e}) \tilde{f}_e, \quad (5)$$

since a non-cut set E cannot contain any bridge. Now consider an outaged line $e \in E$ and a remaining line \hat{e} such that e and \hat{e} are not in the same cell. Theorem 2 implies that $K_{\hat{e}e} = 0$ and therefore $\Delta f_{\hat{e}} = 0$ if e were the *only* line that has been tripped. This does not, however, automatically imply that $K_{\hat{e}e}^E = 0$ for simultaneous line failures E . Indeed, if the tripping of e can impact the mixed flow change $\tilde{f}_{e'}$ on another line $e' \in E$ that is in the same cell as \hat{e} (and hence $K_{\hat{e}e'} \neq 0$), then from (5) we see that $\Delta f_{\hat{e}}$ can potentially be nonzero, resulting in a nonzero $K_{\hat{e}e}^E$. In other words, tripping $e \in E$ in a different cell from \hat{e} can potentially impact \hat{e} through a different simultaneously outaged line $e' \in E$ that is in the same cell as \hat{e} . It is therefore remarkable that tree partitioning turns out to localize the impact of each *simultaneous* line failure within its own cell: the tripping of e can only impact the mixed flow changes $\tilde{f}_{e'}$ on lines $e' \in E$ that are in the same cell, and therefore $K_{\hat{e}e'} \neq 0$ is not possible. This is made precise in the next result.

We collect edges in E based on the cells they belong to and write $E = E_1 \cup E_2 \cup \dots \cup E_k$ as its *block decomposition*. That is, $E_i \subset \mathcal{C}_i$ for some cell \mathcal{C}_i in \mathcal{G} , and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ if $i \neq j$. This decomposition is well-defined since a non-cut set E does not contain any bridge and thus any edge in E must belong to a certain cell.

Proposition 14. Let $E = E_1 \cup E_2 \cup \dots \cup E_k$ be its block decomposition and set $m_i = |E_i|$. Then

- 1) Modulo a permutation of rows and columns, the matrix $(I - B_E C_E^T A C_E)^{-1}$ in (4a) is block-diagonal:

$$\begin{bmatrix} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_k \end{bmatrix}$$

where $H_i \in \mathbb{R}^{m_i \times m_i}$ for $i = 1, 2, \dots, k$.

- 2) Under a line susceptances perturbation μ that is absolutely continuous with respect to \mathcal{L}_m , the submatrix H_i consists of strictly nonzero entries μ -almost surely, i.e.,

$$\mu((H_i)_{e_1 e_2} \neq 0) = 1, \quad \forall e_1, e_2 \in \{1, 2, \dots, m_i\}.$$

The proof of this result is presented in Appendix VI. Proposition 14 allows us to decompose (4) corresponding to the block decomposition $E = E_1 \cup E_2 \cup \dots \cup E_k$. More precisely, writing $f_E = [f_{E_i}; i = 1, \dots, k]$, $\tilde{f}_E = [\tilde{f}_{E_i}; i = 1, \dots, k]$, and $D_{\hat{e},E} = [D_{\hat{e},E_1}, \dots, D_{\hat{e},E_k}]$, equation (4) can be rewritten as:

$$\tilde{f}_{E_i} = H_i f_{E_i}, \quad i = 1, \dots, k \quad (6a)$$

$$\Delta f_{\hat{e}}^{E_i} = D_{\hat{e},E_i} \tilde{f}_{E_i}, \quad \hat{e} \in -E, \quad i = 1, \dots, k \quad (6b)$$

$$\Delta f_{\hat{e}} = \sum_{i=1}^k \Delta f_{\hat{e}}^{E_i}, \quad \hat{e} \in -E. \quad (6c)$$

Equation (6a) reveals the localization structure of tree partitioning for simultaneous line outages. It says that the pre-contingency flows f_{E_i} in cell \mathcal{C}_i mix to produce the vector of flow changes \tilde{f}_{E_i} within the *same* cell according to (6a). Each of these flow changes \tilde{f}_{E_i} impacts the remaining lines $\hat{e} \in \mathcal{E} \setminus E$ *separately* according to (6b). Finally, the total power flow change on \hat{e} is the sum of these changes $\Delta f_{\hat{e}}^{E_i}$ according to (6c).

Since $D_{\hat{e},e}$ shares the same sign as $K_{\hat{e},e}$, $D_{\hat{e},e}$ is non-zero “if” and only if e and \hat{e} are within the same cell by Theorem 2 (recall e is not a bridge since E is a non-cut set). It is thus clear that $\Delta f_{\hat{e}} = 0$ for $\hat{e} \notin \bigcup_{i=1}^k \mathcal{C}_i$. For $\hat{e} \in \mathcal{C}_i$, equation (6c) can be further simplified:

$$\begin{aligned} \Delta f_{\hat{e}} &= D_{\hat{e},E_i} H_i f_{E_i} = \sum_{e' \in E_i} D_{\hat{e},e'} \left(\sum_{e \in E_i} (H_i)_{e'e} f_e \right) \\ &= \sum_{e \in E_i} \left(\sum_{e' \in E_i} K_{\hat{e}e'} (1 - D_{e',e'}) (H_i)_{e'e} \right) f_e. \end{aligned}$$

Therefore, the GLODF can be computed as:

$$K_{\hat{e}e}^E = \begin{cases} \sum_{e' \in E_i} K_{\hat{e}e'} (1 - D_{e',e'}) (H_i)_{e'e} & \hat{e} \in \mathcal{C}_i, e \in E_i, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Note that the summation over e' includes e . It suggests that non-cut failures are localized within the corresponding cells. In particular, for a fixed cell \mathcal{C}_i , the flow changes on remaining lines only come from line failures in E_i , while being independent of any failures from other cells. A complete proof of Theorem 13 is presented in Appendix VII.

B. Cut Failure

Next we consider the case where E forms a cut set, whose removal disconnects \mathcal{G} into multiple islands that are potentially power imbalanced. In this case, the formula (2) no longer applies since the matrix $I - B_E C_E^T A C_E$ is not invertible. The power balancing rule \mathcal{R} then adjusts the injections of participating buses to cancel out the imbalances in all such islands. For ease of presentation, let us focus on one island \mathcal{D} thus created and denote its injection adjustment under \mathcal{R} by $\Delta p_{\mathcal{D}}$. This adjustment $\Delta p_{\mathcal{D}}$ has non-zero components only at participating buses or buses that are endpoints of edges in E that connects \mathcal{D} and $\mathcal{E} \setminus \mathcal{D}$.

Given the fixed island \mathcal{D} , let $E_{\mathcal{D}}$ be the set of edges in E that have both endpoints within this island. Note that $E_{\mathcal{D}}$ is a non-cut set of \mathcal{D} since otherwise tripping E would disconnect \mathcal{D} to multiple islands. Let $B_{\mathcal{D}}, C_{\mathcal{D}}, A_{\mathcal{D}}$ be as introduced in Section V, which correspond to the post-contingency topology of island \mathcal{D} . Let $K^{E_{\mathcal{D}}}$ denote the GLODF of line failure $E_{\mathcal{D}}$ for island \mathcal{D} . With all these notations, we characterize how Δf is related to $\Delta p_{\mathcal{D}}$, $K^{E_{\mathcal{D}}}$ and $E_{\mathcal{D}}$.

From (4), we know that when multiple lines are tripped from the grid simultaneously, the aggregate impact in general is different from the sum of tripping the lines separately. The impact from balancing rule \mathcal{R} , though, turns out to be separable from the rest, as the following result shows (see Appendix VIII for its proof).

Proposition 15. *Given the injection adjustment $\Delta p_{\mathcal{D}}$ and original flow $f_{E_{\mathcal{D}}}$ on $E_{\mathcal{D}}$, we have*

$$\Delta f_{\mathcal{D}} = B_{\mathcal{D}} C_{\mathcal{D}}^T A_{\mathcal{D}} \Delta p_{\mathcal{D}} + K^{E_{\mathcal{D}}} f_{E_{\mathcal{D}}}. \quad (8)$$

The first term in (8) captures the impact of power imbalance, and is characterized by our discussion in Section V-B. The second term in (8) reduces to the case studied in Section VI-A since $E_{\mathcal{D}}$ is a non-cut set of \mathcal{D} . We thus see that a cut set failure impacts the branch flows on remaining lines in two independent ways: (a) via participating buses to distribute the power imbalances; (b) via cells to mix and propagate original flows on the tripped lines. The formula (8) precisely describes the impact propagation through these two ways, which are fully characterized by our results in previous sections.

C. Localization Horizon

In this section, we summarize the results from Sections VI-A and VI-B and show that tree partition localizes the impact of line failures until the grid is disconnected into multiple islands. More formally, given a cascading failure process described by $\mathcal{B}(\mathbf{n})$, $\mathbf{n} \in \mathfrak{N}$, define

$$\mathfrak{T} := \min \{ \mathbf{n} \in \mathfrak{N} : \mathcal{F}(\mathbf{n}) \text{ is a cut set of } \mathcal{G}(\mathbf{n}) \}$$

to be the first stage where the grid is disconnected into multiple islands. Without loss of generality, assume the initial failure $\mathcal{B}(1)$ contains only one edge that belongs to the cell \mathcal{C} . Then we know that (see Appendix IX for a proof):

Proposition 16. *For any $\mathbf{n} \leq \mathfrak{T}$, we have*

$$\mathcal{F}(\mathbf{n}) \subset \mathcal{C}.$$

In other words, in a cascading failure process, the *only* way that a non-bridge failure can propagate to edges outside the cell to which the original failure belongs is to disconnect the grid into multiple islands.

VII. CASE STUDIES

Our findings highlight a new approach for improving the robustness of power grids. More specifically, Theorems 2, 13 and the discussions in Sections IV and VI suggest that it is possible to localize failure propagation by switching off some transmission lines to create a finer tree partition with smaller regions where failures can be contained.

It is reasonable to expect that this approach to improve system robustness *always* comes at the price of increased line congestion. In this section, we argue that this is not necessarily the case, and show that if the lines to switch off are selected properly, it is possible to *improve robustness and reduce congestion simultaneously*. We corroborate this claim using the IEEE 118-bus test network according to two metrics: (a) The *influence graph* is much sparser after tree partitioning, confirming failure localization; (b) The congestion level is reduced on 52.59% of the lines and is increased by less than 10% on 90% of the remaining lines after tree partitioning.

A. Influence Graph

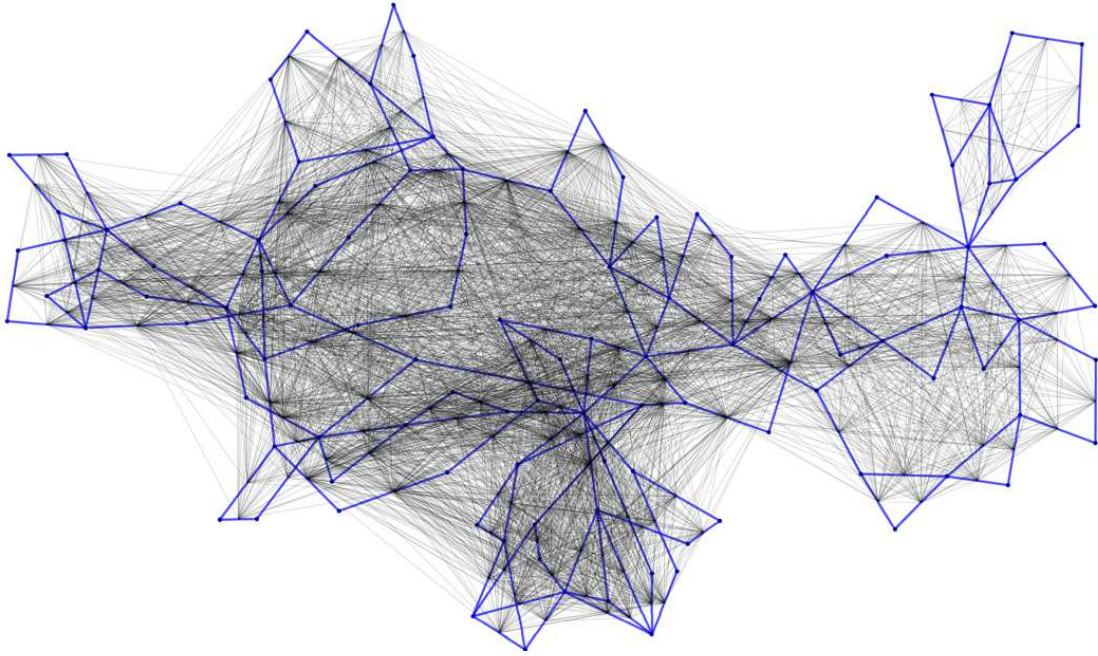
In our experiments, the system parameters are taken from the Matpower Simulation Package [26] and we plot the influence graphs among the transmission lines to demonstrate how a line failure propagates in this network. Our influence graph is similar in concept to that of [27], [28], but unlike their influence graph that is based on a probabilistic model, ours is simply a visualization of the LODF K_{ee} that describes the impact on other lines of tripping a single line e . The original IEEE 118-bus network is depicted in blue in Fig. 3(a).⁵ Its influence graph has these transmission lines (in blue) as its nodes. Two such blue edges e and \hat{e} are connected in the influence graph by a grey edge if $|K_{ee}| \geq 0.005$. As Fig. 3(a) shows, the influence graph of the original network is very dense and connects many edges that are topologically far away, showing a high risk of non-local failure propagation in this network.

We then switch off three transmission lines (indicated as e_1 , e_2 and e_3 in Fig. 3(b)) to obtain a new network topology with a tree partition consisting of two regions of comparable sizes. The corresponding influence graph is shown in Fig. 3(b). Compared to the original influence graph in Fig. 3(a), the new one is much less dense and, in particular, there are no edges connecting transmission lines that belong to different tree-partition regions, as our theory predicts. The network in Fig. 3(b) contains two cut vertices (indicated by c_1 and c_2 in the figure, with c_2 being created when we switch off the lines). It can be checked that line failures are “blocked” by these cut vertices, which verifies our results in Section IV.

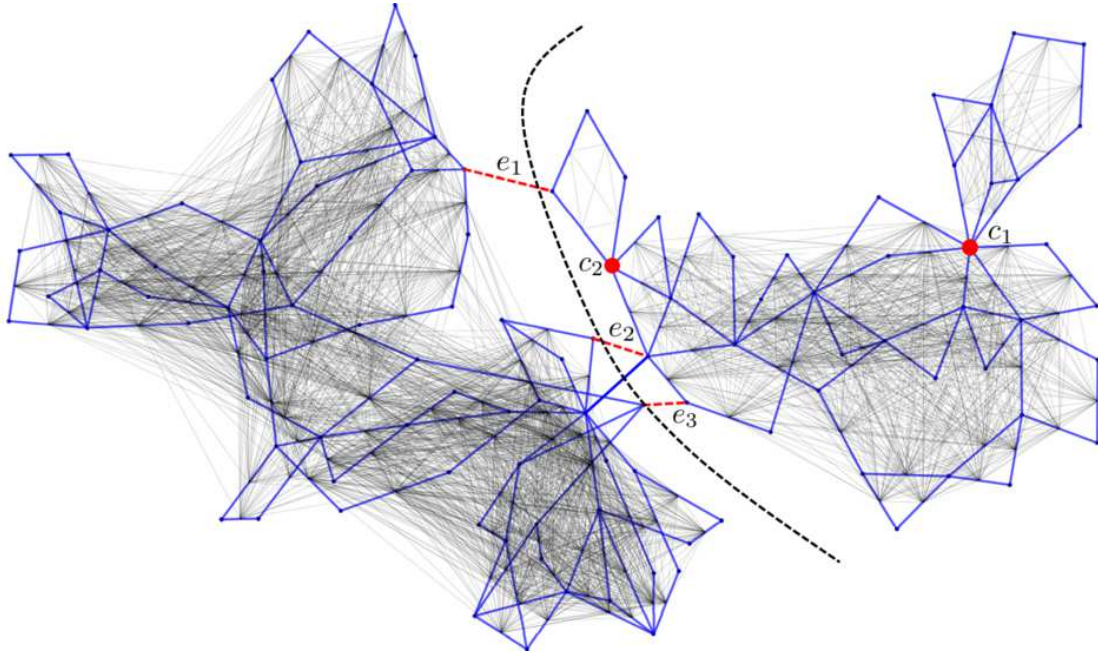
B. Congestion Level

We have demonstrated using a stylized example in Section V of Part I that, contrary to conventional wisdom, tree partitioning can potentially reduce circulating flows in the system and thus decrease the overall line congestion. Here we evaluate changes in line congestion in the IEEE 118-bus network before and after our tree partitioning. Specifically, we collect statistics on the difference between the

⁵The original IEEE 118-bus network has some trivial “dangling” bridges that we remove (collapsing their injections to the nearest bus) to obtain a neater influence graph.



(a) Original IEEE 118-bus network (with edges in blue) and its influence graph (with edges in grey).



(b) The influence graph after switching off e_1 , e_2 and e_3 . The black dashed line indicates the failure propagation boundary defined by the tree-partition. The vertices c_1 and c_2 are cut vertices.

Fig. 3. Influence graphs on the IEEE 118-bus test network before and after switching off lines e_1 , e_2 and e_3 . Blue edges represent physical transmission lines and grey edges represent connections in the influence graph.

branch flows in Fig. 3(b) and those in 3(a), normalized by the original branch flow in Fig. 3(a). They are plotted in Fig. 4(a), showing that 52.59% of the transmission lines have lower congestion. Of the 47.41% of lines that have higher congestion, the majority of these branch flow increases are negligible. To visualize more clearly, we plot in Fig. 4(b) the cumulative distribution function of the normalized positive branch flow changes. It shows that 90% of the branch flows increase by no more than 10%.

VIII. CONCLUSION

We make use of the spectral representation of power redistribution developed in Part I to provide a precise analytical characterization of line failure localizability through tree partitioning. We demonstrate that a tree partition can be computed in linear time. A case study on the IEEE 118-bus test network shows that switching off certain transmission lines can improve grid robustness against line failures without significantly increasing line congestion.

Despite the benefits, refining the tree partition of a given power grid may not be a satisfactory solution by itself in

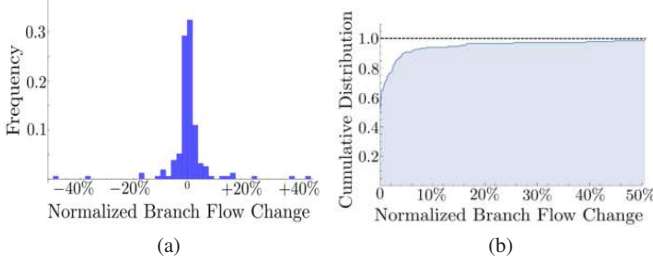


Fig. 4. (a) Histogram of the normalized branch flow changes. (b) Cumulative distribution function of the positive normalized branch flow changes. Note that the curve intercepts the y -axis since 52.59% of the branch flows decrease.

mitigating failures for two reasons. First, switching off lines may lead to single-line vulnerabilities (as demonstrated by the newly created bridge in Fig. 3(b)), whose failure has a global impact on the whole system and can cause significant load loss. Second, tree partitioning is a static construct that does not make use of information on unfolding cascading failures for mitigation. In Part III of this work, we propose a fast timescale control strategy that leverages both the tree partition and a new approach for frequency control to provide guarantees in failure mitigation. The properties of tree partitions derived in this part will be an important building block to the proposed control strategy.

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APPENDIX I PROOF OF PROPOSITION 3

If \hat{e} is a bridge, then since no simple loop can contain a bridge, the Simple Loop Criterion implies that $K_{\hat{e}\hat{e}} = 0$.

If \hat{e} is not a bridge, without loss of generality, we can assume e is within \mathcal{N}_1 and \hat{e} is within \mathcal{N}_2 . Since \mathcal{P} is a tree-partition, any path starting from a node in \mathcal{N}_1 and ending at a node in \mathcal{N}_2 must pass through all bridges in the path in the reduced graph $\mathcal{G}_{\mathcal{P}}$ (see Part I for more details of this reduced graph) from \mathcal{N}_1 to \mathcal{N}_2 . As a result, any loop containing both e and \hat{e} must pass through these bridges at least twice, and thus is not simple. By the Simple Loop Criterion, we then know that $K_{\hat{e}\hat{e}} = 0$. \square

APPENDIX II PROOF OF PROPOSITION 5

Let \mathcal{C} be the cell that e and \hat{e} belong to and write $e = (i, j)$ and $\hat{e} = (w, z)$. Consider the polynomial $f(B)$ with line susceptances $B = (B_e : e \in \mathcal{E})$ as variables given by

$$f(B) := \sum_{E \in \mathcal{T}(\{i, w\}, \{j, z\})} \chi(E) - \sum_{E \in \mathcal{T}(\{i, z\}, \{j, w\})} \chi(E).$$

We claim that f is not identically zero.

First, let C be a simple cycle in \mathcal{C} that contains both e and \hat{e} . Such a cycle always exists as \mathcal{C} is 2-connected by construction, and it is a classical result that any pair of edges are contained in a simple cycle for 2-connected graphs [18]. Therefore, we can find two disjoint paths P_1

and P_2 connecting the endpoints of e and \hat{e} . Without loss of generality, assume P_1 connects i to w and P_2 connects j to z . By iteratively adding edges from \mathcal{G} to P_1 and P_2 , we can extend P_1 and P_2 to a spanning forest of \mathcal{G} consisting of exactly two trees. Moreover, the tree extended from P_1 contains $\{i, w\}$ and the tree extended from P_2 contains $\{j, z\}$. Thus we have constructed an element of $\mathcal{T}(\{i, w\}, \{j, z\})$. Denote this element by E_0 .

Second, we show that

$$\mathcal{T}(\{i, w\}, \{j, z\}) \cap \mathcal{T}(\{i, z\}, \{j, w\}) = \emptyset.$$

Indeed, consider an element E_1 from $\mathcal{T}(\{i, w\}, \{j, z\})$, which consists of two trees \mathcal{T}_1 and \mathcal{T}_2 with \mathcal{T}_1 containing $\{i, w\}$ and \mathcal{T}_2 containing $\{j, z\}$. If $E_1 \in \mathcal{T}(\{i, z\}, \{j, w\})$, then \mathcal{T}_1 must also contain z . However, this implies $z \in \mathcal{T}_1 \cap \mathcal{T}_2$, and thus \mathcal{T}_1 and \mathcal{T}_2 are not disjoint, contradicting the definition of $\mathcal{T}(\{i, w\}, \{j, z\})$.

As a result, we see that the element E_0 constructed in our first step contributes a term to $\sum_{E \in \mathcal{T}(\{i, w\}, \{j, z\})} \chi(E)$ but not to $\sum_{E \in \mathcal{T}(\{i, z\}, \{j, w\})} \chi(E)$. Therefore $f(B)$ contains non-vanishing terms and is not identically zero.

It is well-known from algebraic geometry that the root set of a polynomial which is not identically zero has Lebesgue measure zero [29]. That is, we have

$$\mu(f(B + \omega) = 0) = \mathcal{L}_m(f(B + \omega) = 0) = 0,$$

where the first equality is because μ is absolutely continuous with respect to \mathcal{L}_m (it is clear that the root set of the polynomial f is measurable since f is continuous).

Finally, by Theorem 1, $K_{\hat{e}e} = 0$ if and only if $f(B + \omega) = 0$. This fact implies that

$$\mu(K_{\hat{e}e} \neq 0) = 1 - \mu(K_{\hat{e}e} = 0) = 1 - \mu(f(B + \omega) = 0) = 1$$

and completes our proof. \square

APPENDIX III

PROOF OF PROPOSITION 8

For simplicity, we drop the subscript \mathcal{D} for all related quantities. Let $M := f_e$ be the total amount of power shortfall (excess if M is negative) in island \mathcal{D} after the bridge failure. Under the balancing rule \mathcal{R} , we can calculate the power injection adjustment Δp as follows:

$$\Delta p_j = \begin{cases} \alpha_j M, & \text{if } j \in \mathcal{N}_{\mathcal{R}} \\ -\sum_{k \in \mathcal{N}_{\mathcal{R}}, k \neq u} \alpha_k M, & \text{if } j = u \\ 0, & \text{otherwise.} \end{cases}$$

Let u be the slack bus in \mathcal{D} , the flow change on edge \hat{e} can be computed as:

$$\Delta f_{\hat{e}} = B_{\hat{e}} C_{\hat{e}}^T A \Delta p = M \sum_{j \in \mathcal{N}_{\mathcal{R}}} D_{\hat{e}, ju} \alpha_j,$$

where $D_{\hat{e}, ju}$ is the generation shift sensitivity factor (see details in Part I). Note that $D_{\hat{e}, uu} = 0$.

Write $\hat{e} = (w, z)$. For any $j \in \mathcal{N}_{\mathcal{R}}$, let $g^j(B)$ be the following polynomial in line susceptances ($B_e : e \in \mathcal{E}$):

$$\sum_{E \in \mathcal{T}_{\mathcal{E}}(\{j, w\}, \{u, z\})} \chi(E) - \sum_{E \in \mathcal{T}_{\mathcal{E}}(\{j, z\}, \{u, w\})} \chi(E),$$

where $\mathcal{T}_{\mathcal{E}}(\mathcal{N}_1, \mathcal{N}_2)$ is the set of spanning forests of $\mathcal{G}_{\mathcal{D}}$ consisting of exactly two trees that contain \mathcal{N}_1 and \mathcal{N}_2 respectively. By our graphical representation of the generation

shift sensitivity factor from Part I, the branch flow change on \hat{e} is given by

$$\Delta f_{\hat{e}} = \frac{B_{\hat{e}} M}{\sum_{E \in \mathcal{T}_{\mathcal{E}}} \chi(E)} \sum_{j \in \mathcal{N}_{\mathcal{R}}} \alpha_j g^j(B).$$

Let $g(B) := \sum_{j \in \mathcal{N}_{\mathcal{R}}} \alpha_j g^j(B)$. If $\Delta f_{\hat{e}} \neq 0$, then we can find at least one j such that $\mathcal{T}_{\mathcal{E}}(\{j, w\}, \{u, z\})$ or $\mathcal{T}_{\mathcal{E}}(\{j, z\}, \{u, w\})$ is nonempty. Without loss of generality, assume that $\mathcal{T}_{\mathcal{E}}(\{j, w\}, \{u, z\})$ is nonempty. Any element in $\mathcal{T}_{\mathcal{E}}(\{j, w\}, \{u, z\})$ contains two trees containing $\{j, w\}$ and $\{u, z\}$ respectively, and thus induces one path from w to j and another path from u to z . By adjoining (w, z) to these two paths, we can create a path from j to u that contains \hat{e} . Therefore, $\Delta f_{\hat{e}} \neq 0$ only if there exists $j \in \mathcal{N}_{\mathcal{R}}$ such that there is a path in \mathcal{D} from u to a participating bus j containing \hat{e} .

On the other hand, if there is a simple path P in \mathcal{D} from u to a participating bus j containing \hat{e} , we claim $g(B)$ is not identically zero. Indeed, by a similar argument to the proof of Proposition 5, we know that for any $i, j \in \mathcal{N}_{\mathcal{R}}$ (including the case $i = j$), the following is true:

$$\mathcal{T}_{\mathcal{E}}(\{i, w\}, \{u, z\}) \cap \mathcal{T}_{\mathcal{E}}(\{j, z\}, \{u, w\}) = \emptyset.$$

As a result, a term in $g^i(B)$ with positive coefficient is never canceled by a term in $g^j(B)$ with negative coefficient, and vice-versa. Therefore, to show $g(B)$ is not identically zero, it suffices to show at least one term of $g^j(B)$ is not identically zero.

Given the the graph $\mathcal{G}_{\mathcal{D}}$ is connected, we can extend the path P to a spanning tree of $\mathcal{G}_{\mathcal{D}}$. By removing edge \hat{e} , we can create a tree from u to one end point of \hat{e} , say w , and another tree from j to z . In particular, these two trees form exactly a spanning forest of $\mathcal{G}_{\mathcal{D}}$, which is an element of $\mathcal{T}_{\mathcal{E}}(\{j, z\}, \{u, w\})$. This contributes to a term in $g^j(B)$ that is not identically zero.

Again by the classical algebraic geometry result asserting the root set of any polynomial that is not identically zero has Lebesgue measure zero [29], and from the absolute continuity of μ it follows that

$$\mu(\Delta f_{\hat{e}} = 0) = \mathcal{L}_m(g(B + \omega) = 0) = 0$$

and thus $\mu(\Delta f_{\hat{e}} \neq 0) = 1$. \square

APPENDIX IV

PROOF OF LEMMA 10

Let \mathcal{C} be the cell that contains \hat{e} . Since \mathcal{N}_i is a participating region, we know there exists a bus within \mathcal{C} , say n_1 , that participates the power balance and is not a cut vertex. Recall that any path from u to \mathcal{C} must go through a common cut vertex in \mathcal{C} [18], say n_e . Now by adding an edge between n_e and n_1 (if such edge did not originally exist), the resulting cell \mathcal{C}' is still 2-connected. Thus there exists a simple loop in \mathcal{C}' that contains the edge (n_e, n_1) and $\hat{e} = (w, z)$, which implies we can find two disjoint paths P_1 and P_2 connecting the endpoints of these two edges. Without loss of generality, assume P_1 connects n_e to w and P_2 connects n_1 to z . By concatenating the path from u to n_e , we can extend P_1 to a path \tilde{P}_1 from u to w , which is still disjoint from P_2 . Now, by adjoining \hat{e} to \tilde{P}_1 and P_2 , we can construct a path from u to n_1 that passes through \hat{e} . Proposition 8 then implies $\mu(\Delta f_{\hat{e}} \neq 0) = 1$. \square

APPENDIX V
PROOF OF PROPOSITION 11

If \hat{e} is a bridge, denote the two connected components of \mathcal{D} after removing \hat{e} as \mathcal{D}_1 and \mathcal{D}_2 , and without loss of generality assume \mathcal{D}_1 is originally connected to e in \mathcal{G} . It is easy to see that the branch flow change on \hat{e} is given by

$$\Delta f_{\hat{e}} = \sum_{j \in \mathcal{D}_2} (\Delta p_{\mathcal{D}})_j \neq 0,$$

where the last \neq is because the grid is participating and thus all tree-partition regions in \mathcal{D}_2 would adjust their injections (in the same “direction” as α_j ’s are positive).

If \hat{e} is not a bridge, then Lemma 10 implies $\mu(\Delta f_{\hat{e}} \neq 0) = 1$. \square

APPENDIX VI
PROOF OF PROPOSITION 14

Let $L^E := C_E^T A C_E$. First we claim that if $e_i \in E_i$, $e_j \in E_j$ and $i \neq j$, then $L_{e_i e_j}^E = 0$. To see this, note that

$$(B_E L^E)_{e_i e_j} = D_{e_i, e_j},$$

where D_{e_i, e_j} is the generation shift sensitivity factor from e_j to e_i (see Part I). It is easy to see that $D_{e_i, e_j} = 0$ if and only if $K_{e_i e_j} = 0$. As a result, if e_i and e_j are in different cells, then $D_{e_i, e_j} = 0$.

Since I is diagonal, by permuting the edges according to the cells they belong to, the matrix $I - B_E L^E$ has the following block-diagonal form:

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix},$$

where $J_i \in \mathbb{R}^{m_i \times m_i}$ for $i = 1, 2, \dots, k$. Moreover, each J_i is invertible since $I - B_E L^E$ is invertible (see [22] for instance). This, in particular, implies that $(I - B_E L^E)^{-1}$ is of the form

$$\begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_k \end{bmatrix},$$

where $H_i \in \mathbb{R}^{m_i \times m_i}$ for $i = 1, 2, \dots, k$.

Next, we consider H_1 and fix $e_1, e_2 \in \{1, 2, \dots, m_1\}$ to be two edges in the cell \mathcal{C}_1 . Then,

$$(H_1)_{e_1 e_2} = \frac{\det(J_1^{e_1, e_2})}{\det(J_1)},$$

where $J_1^{e_1, e_2}$ is the matrix obtained from J_1 by replacing the e_1 -th column with a vector with value 1 at e_2 -th component and 0 elsewhere. Set J'_1 to be the submatrix of J_1 obtained by removing the e_1 -th column and e_2 -th row from J_1 . Then,

$$\det(J_1^{e_1, e_2}) = (-1)^{e_1 + e_2} \det(J'_1).$$

The entries of J'_1 are of the form $1 - D_{e_i, e_i}$ or D_{e_i, e_j} . Recall from Part I that for any pair of edges $e_i = (u, v)$, $e_j = (w, z)$, we have D_{e_i, e_j} be given by

$$\frac{B_{e_i}}{\sum_{F \in \mathcal{T}_{\mathcal{E}}} \chi(F)} \times \left(\sum_{F \in \mathcal{T}(\{u, w\}, \{v, z\})} \chi(F) - \sum_{F \in \mathcal{T}(\{u, z\}, \{v, w\})} \chi(F) \right).$$

Note that we use F to indicate the edge set of spanning forests, and set E as the set of line failures. Since all edges in E_1 are within the same cell, D_{e_i, e_j} is not identically zero for any $e_i, e_j \in E_1$ (including the case $e_i = e_j$). We now show that $\det(J'_1)$ is a polynomial in B that is not identically zero. To do so, we first prove the following three technical lemmas:

Lemma 17. *If $e_i \neq e_j$ are two different edges, then any term in the numerator of D_{e_i, e_j} does not contain B_{e_j} .*

Proof. The numerator of D_{e_i, e_j} is given by the difference of

$$B_{uv} \sum_{F \in \mathcal{T}(\{u, w\}, \{v, z\})} \chi(F)$$

and

$$B_{uv} \sum_{F \in \mathcal{T}(\{u, z\}, \{v, w\})} \chi(F).$$

When $e_i \neq e_j$, for each element of $F \in \mathcal{T}(\{u, w\}, \{v, z\})$, adding $e_i = (u, v)$ to F induces a spanning tree of \mathcal{G} that passes through $e_i = (u, v)$ but not $e_j = (w, z)$, and $d_F(u, w) < d_F(v, w)$, where $d_F(\cdot, \cdot)$ means the distance in terms of minimum number of hops in F . Conversely, for any spanning tree of \mathcal{G} that passes through $e_i = (u, v)$ but not $e_j = (w, z)$, and $d_F(u, w) < d_F(v, w)$, removing e_i induces an element of $F \in \mathcal{T}(\{u, w\}, \{v, z\})$.

A similar argument also applies to $\mathcal{T}(\{u, z\}, \{v, w\})$. Therefore, the numerator of D_{e_i, e_j} is exactly given as

$$\sum_{F \in \mathcal{T}_{e_i, -e_j}} \text{sign}(F) \chi(F), \quad (9)$$

where $\mathcal{T}_{e_i, -e_j}$ is the set of spanning trees of \mathcal{G} that pass through $e_i = (u, v)$ but not $e_j = (w, z)$, and

$$\text{sign}(F) := \begin{cases} 1, & d_F(u, w) < d_F(v, w) \\ -1, & d_F(u, w) \geq d_F(v, w). \end{cases}$$

In particular, B_{e_j} does not appear in any of these terms. \square

Lemma 18. *If $e_i = e_j$, then the numerator of D_{e_i, e_j} is*

$$\sum_{F \in \mathcal{T}_{e_i}} \chi(F),$$

where \mathcal{T}_{e_i} consists of all spanning trees of \mathcal{G} that pass through e_i .

Proof. When $e_i = e_j$, $D_{e_i, e_j}/B_{e_i}$ reduces to the effective reactance of e_i , and the result then follows directly from Corollary 7 of Part I. \square

Lemma 19. *Let g_1, g_2, \dots, g_{l_1} and h_1, h_2, \dots, h_{l_2} be functions in B of the form D_{e_i, e_j} with $e_i, e_j \in \mathcal{C}_1$. Assume the e_j ’s for g_k are different over $k = 1, 2, \dots, l_1$, and the e_j ’s for h_k are different over $k = 1, 2, \dots, l_2$. Let q_1 be the number of g_k ’s with $e_i = e_j$, and q_2 be the number of h_k ’s with $e_i = e_j$. If $q_1 \neq q_2$, then for any fixed $a_1, a_2 \neq 0$, the function*

$$f(B) := a_1 \prod_{k=1}^{l_1} g_k + a_2 \prod_{k=1}^{l_2} h_k$$

is not identically zero.

Proof. Without loss of generality, assume $l_1 \leq l_2$ and $q_1 < q_2$. Set $\zeta(B) = \sum_{E \in \mathcal{T}_E} \chi(E)$, by collecting a common denominator, we then see that

$$f(B) = \frac{a_1 \zeta^{(l_2-l_1)}(B) \tilde{g}(B) + a_2 \tilde{h}(B)}{\zeta^{l_2}(B)},$$

where $\tilde{g}(B)$ and $\tilde{h}(B)$ are homogeneous polynomials in B with order $l_1(n-1)$ and $l_2(n-1)$, respectively.

Let $\alpha(g) := (e_j : g_k = D_{e_i, e_j}, e_i \neq e_j, k = 1, 2, \dots, l_1)$ be the vector collecting all edges e_j corresponding to terms in g_k 's with $e_i \neq e_j$, and define $\alpha(h)$ similarly. Since $q_1 < q_2$, we can find an edge \tilde{e}_j in $\alpha(h)$ that is not in $\alpha(g)$. Without loss of generality, say this specific h is $h_{\tilde{k}}$. By Lemma 17, we know that the numerator of $h_{\tilde{k}}$ does not contain $B_{\tilde{e}_j}$. As a result, the order of $B_{\tilde{e}_j}$ is at most $l_2 - 1$ in all terms of $\tilde{h}(B)$.

Now, we claim that $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$ contains a term where $B_{\tilde{e}_j}$ is of order l_2 , which is strictly larger than $l_2 - 1$. This term cannot be canceled by any term from $\tilde{h}(B)$, and thus we know $f(B)$ is not identically zero. To show this claim, note that by expanding $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$, we see that each term in $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$ is a product of terms from the factors involved. Thus it suffices to pick a term that contains $B_{\tilde{e}_j}$ from each of these factors.

First we consider the ζ factor. Since ζ sums over all spanning trees in \mathcal{G} , and any edge in \mathcal{G} can be extended to a spanning tree of \mathcal{G} by iteratively adding edges, we know there exists at least one term in ζ that contains $B_{\tilde{e}_j}$. We pick this term out for every ζ in $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$, which multiplies to a term in which $B_{\tilde{e}_j}$ has order $l_2 - l_1$.

Next, we consider g_k 's with $e_i = e_j$. For such g_k , since e_i and \tilde{e}_j are within the same cell, we can find a simple loop in this cell that contains both e_i and \tilde{e}_j . Removing an edge different from e_i and \tilde{e}_j from this loop induces a path which, by iteratively adding edges, can be extended to a spanning tree that passes through e_j and \tilde{e}_j . This tree is an element of \mathcal{T}_{e_i} , and thus by Lemma 18 appears in the numerator of g_k . We pick this term for g_k , which contains $B_{\tilde{e}_j}$. Doing so for all g_k 's of this type contributes $l_1 - q_1$ order of $B_{\tilde{e}_j}$.

Lastly, we consider g_k 's with $e_i \neq e_j$. Since e_i and \tilde{e}_j are within the same cell, we can find a simple loop in this cell that contains both e_i and \tilde{e}_j . If e_j is also in this loop, then we remove e_j . Otherwise, remove an edge different from e_i and \tilde{e}_j . This induces a path that contains e_i and \tilde{e}_j but not e_j . By iteratively adding edges other than e_j , we can extend this path into a spanning tree of \mathcal{G} that contains e_i and \tilde{e}_j but not e_j , which in particular is an element of $\mathcal{T}_{e_i, -e_j}$. By (9) we then know that this tree appears in the numerator of g_k . We pick this term for g_k , which contains $B_{\tilde{e}_j}$. Doing so for all g_k 's of this type contributes q_1 order of $B_{\tilde{e}_j}$.

In summary, we can find a term that contains $B_{\tilde{e}_j}$ in every factor of $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$. Multiplying all these terms together induces a term of $\zeta^{(l_2-l_1)}(B) \tilde{g}(B)$ where $B_{\tilde{e}_j}$ is of order l_2 . Our claim then follows, and this completes the proof. \square

Recall that

$$\det(J'_1) = \sum_{\sigma \in \mathcal{S}_{m_1-1}} \text{sgn}(\sigma) \prod_e (J'_1)_{e\sigma(e)}, \quad (10)$$

where \mathcal{S}_{m_1-1} is the symmetric group of order $m_1 - 1$ and $\text{sgn}(\sigma)$ is the signature of σ .

Now, depending on whether $e_1 = e_2$, the matrix J'_1 contains either $m_1 - 1$ or $m_1 - 2$ entries of the form $1 - D_{\tilde{e}_i, \tilde{e}_j}$. When there are $m_1 - 1$ such entries, they must all appear on the diagonal of J'_1 , and thus multiplying all these terms induces a term in (10) that has the form

$$\prod_{k=1}^{m_1-1} D_{e_{i_k}, e_{i_k}}.$$

All other terms in $\det(J'_1)$ contain at least one factor of the form $D_{\tilde{e}_i, \tilde{e}_j}$ with $\tilde{e}_i \neq \tilde{e}_j$ (and hence at most $m_1 - 2$ factor of the form $D_{\tilde{e}_i, \tilde{e}_i}$), and thus by Lemma 19, cannot cancel the above term. As a result, we know that $\det(J'_1)$ is not identically zero.

When there are $m_1 - 2$ entries of the form $1 - D_{\tilde{e}_i, \tilde{e}_j}$ in J'_1 , these entries must appear on the diagonal of J'_1 . And thus multiplying them together with the remaining diagonal entry of J'_1 induces a term in (10) that has exactly $m_1 - 2$ factors of the form $D_{\tilde{e}_i, \tilde{e}_i}$. Meanwhile, all other terms in $\det(J'_1)$ contain at least two factors of the form $D_{\tilde{e}_i, \tilde{e}_j}$ with $\tilde{e}_i \neq \tilde{e}_j$ (and hence at most $m_1 - 3$ factors of the form $D_{\tilde{e}_i, \tilde{e}_i}$), and thus by Lemma 19, cannot cancel the above term. As a result, $\det(J'_1)$ is not identically zero.

In summary, we have shown that $\det(J'_1)$ is a rational function that is not identically zero. Therefore,

$$\mu(\det(J'_1) = 0) = \mathcal{L}_m(\det(J'_1) = 0) = 0,$$

or, equivalently, $(H_1)_{e_1 e_2} \neq 0$ almost surely in μ . \square

APPENDIX VII

PROOF OF THEOREM 13

Without loss of generality, assume $e \in E_1$. Proposition 14 implies that the e -th column of $B_{-E} C_{-E}^T A C_E (I - B_E C_E^T A C_E)^{-1}$ is given by

$$B_{-E} C_{-E}^T A C_{E_1} (H_1)_e,$$

where $(H_1)_e$ is the column of H_1 corresponding to the edge e . It then follows that

$$K_{ee}^E = \sum_{e' \in E_1} D_{\tilde{e}, e'} (H_1)_{e' e}.$$

From the proof of Proposition 14, we can rewrite the above formula to

$$K_{ee}^E = \frac{\sum_{e' \in E_1} D_{\tilde{e}, e'} \det(J_1^{e', e})}{\det(J_1)}, \quad (11)$$

where $J_1^{e', e}$ is the matrix obtained from J_1 by replacing the e' -th column with a vector with value 1 at e -th component and 0 elsewhere.

If $\hat{e} \notin \mathcal{C}_1$, then $D_{\tilde{e}, e'} = 0$ for all $e' \in E_1$ by Theorem 2. Thus we know $K_{ee}^E = 0$.

If $\hat{e} \in \mathcal{C}_1$, then we claim that the numerator of (11) is not identically zero. If the claim is true, then

$$\mu(K_{ee}^E \neq 0) = \mathcal{L}_m(K_{ee}^E \neq 0) = 1$$

and this completes the proof.

We now show the claim indeed holds. First we consider the case $\hat{e} \notin E_1$. In this case, note that the term in (11) corresponding to $e' = e$ contains one term that has exactly $m_1 - 1$ factors of the form $D_{\tilde{e}, \tilde{e}}$ ($\hat{e} \neq e'$ for all $e' \in E_1$ in this case), while all other terms in the numerator of (11) contains

at most $m_1 - 2$ factors of the form $D_{\hat{e}, \hat{e}}$. From Lemma 19 we then see that the numerator of (11) is not identically zero.

Next we consider the case $\hat{e} \in E_1$ but $\hat{e} \neq e$. In this case, among all terms of (11), only the terms corresponding to $e' = \hat{e}$ and $e' = e$ contain one term that has exactly $m_1 - 1$ factors of the form $D_{\hat{e}, \hat{e}}$, and these two terms are

$$D_{\hat{e}, e} D_{\hat{e}, \hat{e}} \prod_{e'' \neq e, \hat{e}} D_{e'', e''}$$

and

$$D_{e, \hat{e}} D_{\hat{e}, \hat{e}} \prod_{e'' \neq e, \hat{e}} D_{e'', e''},$$

respectively, which do not cancel each other since $\hat{e} \neq e$. All other terms in the numerator of (11) contain at most $m_1 - 2$ factors of the form $D_{\hat{e}, \hat{e}}$. From Lemma 19 we then see that the numerator of (11) is not identically zero.

Finally we consider the case $\hat{e} = e$. In this case, among all terms of (11), only the term corresponding to $e' = e$ contains one term that has exactly m_1 factors of the form $D_{\hat{e}, \hat{e}}$, while all other terms in the numerator of (11) contain at most $m_1 - 2$ factors of the form $D_{\hat{e}, \hat{e}}$. Lemma 19 then implies that the numerator of (11) is not identically zero, which concludes the proof of the claim. \square

APPENDIX VIII

PROOF OF PROPOSITION 15

Denote the original injection over \mathcal{D} by $p_{\mathcal{D}}$, then after tripping E , the new injection in \mathcal{D} is $p_{\mathcal{D}} + \Delta p_{\mathcal{D}}$. Therefore, the new power flow is given by

$$\tilde{f}_{\mathcal{D}} = B_{\mathcal{D}} C_{\mathcal{D}}^T A_{\mathcal{D}} (p_{\mathcal{D}} + \Delta p_{\mathcal{D}}).$$

Since $B_{\mathcal{D}} C_{\mathcal{D}}^T A_{\mathcal{D}} p_{\mathcal{D}}$ is simply the power flow on \mathcal{D} after $E_{\mathcal{D}}$ is tripped under the original injection $p_{\mathcal{D}}$, we see that

$$B_{\mathcal{D}} C_{\mathcal{D}}^T A_{\mathcal{D}} p_{\mathcal{D}} = f_{\mathcal{D}} + K^{E_{\mathcal{D}}} f_{\mathcal{E}_{\mathcal{D}}}.$$

The desired result then follows. \square

APPENDIX IX

PROOF OF PROPOSITION 16

By Theorem 13, for any \mathbf{n} , if $\mathcal{B}(\mathbf{n}) \subset \mathcal{C}$ and $\mathcal{F}(\mathbf{n})$ is a non-cut set of $\mathcal{G}(\mathbf{n})$, then only edges within \mathcal{C} can potentially have a new branch flow, and therefore $\mathcal{F}(\mathbf{n}) \subset \mathcal{C}$. If $\mathcal{F}(\mathbf{n})$ is a non-cut set of $\mathcal{G}(\mathbf{n})$, then $\mathcal{B}(\mathbf{n} + 1) = \mathcal{B}(\mathbf{n}) \cup \mathcal{F}(\mathbf{n})$ is a non-cut set of $\mathcal{G}(\mathbf{n} + 1)$. The desired result thus follows by induction. \square

APPENDIX X

PERTURBATIONS WITH DIRAC MASSES

In our discussions on the perturbation μ , we have assumed μ to be absolutely continuous with respect to the Lebesgue measure \mathcal{L}_m on \mathbb{R}^m . The main purpose of this assumption is to simplify the presentation. In this appendix, we demonstrate for Proposition 5 how our results can be generalized when this absolute continuity assumption is not imposed.

Let $\Omega := \{\omega : f(B + \omega) = 0, \mu(\{\omega\}) > 0\}$ be the set of Dirac masses of μ at which the polynomial f (as defined in the Proof of Proposition 5) vanishes. It is easy to show that Ω is a countable set and hence measurable (thus $\mu(\Omega)$ is meaningful). From the proof of Proposition 5, it is not hard to see the following:

Proposition 20. *Consider a power network \mathcal{G} with susceptibility perturbation μ and let e, \hat{e} be two distinct edges within the same cell. We have:*

$$\mu(K_{\hat{e}e} \neq 0) = 1 - \mu(\Omega).$$

All of our results involving μ can be generalized in this way. \square